Last time:

general outline how to solve 1-dim heat equation

Do another example, Section 2.4.

(PDE) $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

(BC) $$\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$$

(IC) $$u(x,0) = f(x)$$ given function

interpretation of (BC) complete insulation how to solve:

I Find product solutions

$$u(x,t) = G(t)\,\phi(x)$$

Separate variables:

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

$\Rightarrow$ get 2 ODE's

$$G'(t) = -\lambda k G(t)$$

$$\phi''(x) = -\lambda\,\phi(x)$$

solve:

$$G(t) = C e^{-\lambda k t}$$

$$\phi(x) = C_1 \cos\sqrt{\lambda}\,x + C_2 \sin\sqrt{\lambda}\,x$$

for $\lambda \geq 0$

Can show as before for $\lambda < 0$ · no interesting solutions

possible values for λ?

use boundary conditions $\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$

$$\frac{\partial u}{\partial x} = \phi'(x)\, G(t)$$

$$\phi'(x) = -C_1\sqrt{\lambda}\,\sin\sqrt{\lambda}\,x + C_2\sqrt{\lambda}\,\cos\sqrt{\lambda}\,x$$

$$BC \Rightarrow \quad 0 = \phi'(0) = -C_1\sqrt{\lambda}\,\underbrace{\sin 0}_{=0} + C_2\sqrt{\lambda}\,\underbrace{\cos 0}_{=1}$$

$$= C_2\sqrt{\lambda}$$

$$\Rightarrow \boxed{C_2 = 0}$$

$$BC \Rightarrow \quad 0 = \phi'(L) = -C_1\sqrt{\lambda}\,\sin\sqrt{\lambda}\,L$$

$$\Rightarrow \boxed{\sqrt{\lambda}\,L = n\pi, \quad n = 0,1,2\ldots} \quad \left(\begin{array}{l}\sin -nx \\ = -\sin nx\end{array}\right)$$

$\underbrace{\text{as}}_{\text{almost}}$ in previous example

eigenvalues are $\frac{n^2\pi^2}{L^2}$, $n = 0, 1, 2, 3, \ldots$

new

$\Rightarrow \quad \phi(x) = \cos \frac{n\pi}{L} x, \quad n = 0, 1,$

for $n=0$ $\cos 0 = 1 \Rightarrow$ constant solution

$\Rightarrow$ get product solutions

$$\boxed{u(x,t) = \cos \frac{n\pi}{L} x \, e^{-\left(\frac{n\pi}{L}\right)^2 k t} \qquad n = 0, 1, 2 \ldots}$$

Exploit IC

want to write $f(x)$ as a linear combination of $\cos\frac{n\pi}{L}x$, $n=0,1,\ldots$

Theorem 1

$$\int_0^L \cos\frac{n\pi}{L}x \cos\frac{m\pi}{L}x\,dx = \begin{cases} 0 & n\neq m \\ L/2 & n=m>0 \\ L & n=m=0 \end{cases}$$

(aside: if $n=m=0 \Rightarrow$ integral $= \int_0^L 1\,dx = L$)

Theorem 2 If $f(x)$ cont. function on $[0, L]$

$$\Longrightarrow \boxed{f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{L} x}$$

where $$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx \quad n > 0$$

$$A_0 = \frac{1}{L} \int_0^L f(x) \, dx$$

in general: if $u_n = \cos \frac{n\pi}{L} x, \quad n \geq 0$

$$\boxed{A_n = \frac{(f, u_n)}{(u_n, u_n)}}$$

Get from this general solution of our problem.

$$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{L}x \, e^{-\left(\frac{n\pi}{L}\right)^2 kt}}$$

for $n=0$ $\quad A_0 \cos\frac{0\pi}{L}x \, e^{-\left(\frac{0\pi}{L}\right)^2 kt} = A_0$

what happens if $t \to \infty$?

if $n>0$: $\quad \left|\cos\frac{n\pi}{L}x\right| \leq 1$

$$e^{-\left(\frac{n\pi}{L}\right)^2 kt} \to 0$$

$\Bigg\}$ $\sum A_n \cos\frac{n\pi}{L}x \, e^{-(\dots)} \longrightarrow 0$ for $t \to \infty$

$\Rightarrow u(x,t) \longrightarrow A_0$ for $t \to \infty$

Physical interpretation:

Recall that $A_0 = \frac{1}{L} \int_0^L f(x)\,dx$

$= \text{average temperature}$

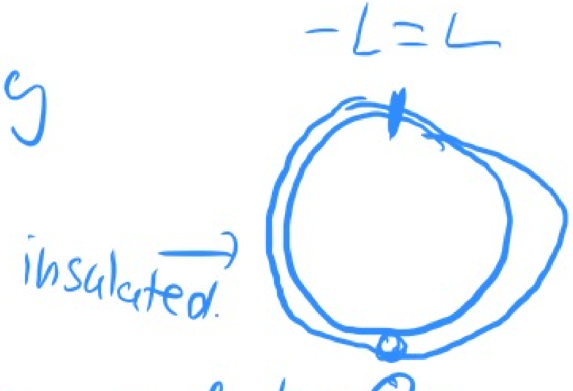
If rod is completely insulated

$\Rightarrow$ temperature will be average temperature every where for $t \to \infty$

So far: $\phi(x)$'s where either all sines
or all cosines.

an example where both sines and cosines can occur:

consider ring with circumference $= 2L$



set up heat equation and boundary conditions:

identify ring with interval
$[-L, L]$

$\Rightarrow$ coordinate $-L$ $\approx$ coordinate L

get following PDE, BC and IC

(PDE) $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$

(IC) $u(x,0) = f(x)$ $\quad -L \le x \le L$

(BC) $$\boxed{\begin{aligned} u(-L,t) &= u(L,t) \\ \frac{\partial u}{\partial x}(-L,t) &= \frac{\partial u}{\partial x}(L,t) \end{aligned}}$$

effect on solution:

as before $u(x,t) = \phi(x)\, G(t)$

$$\boxed{\phi(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x}$$

Determine λ's from BC

$\Rightarrow \quad \phi(-L) = \phi(L) \qquad (1)$

$\phi'(-L) = \phi'(L) \qquad (2)$

(1) $$c_1 \cos\sqrt{\lambda} L + c_2 \sin\sqrt{\lambda} L =$$
$$= c_1 \cos(-\sqrt{\lambda} L) + c_2 \sin(-\sqrt{\lambda} L)$$

(2) $$c_1 \sqrt{\lambda}(-\sin\sqrt{\lambda} L) + c_2 \sqrt{\lambda} \cos\sqrt{\lambda} L$$
$$= c_1 \sqrt{\lambda}(-\sin(-\sqrt{\lambda} L)) + c_2 \sqrt{\lambda}(\cos(-\sqrt{\lambda} L))$$

$\Rightarrow$ functions need to be periodic wite period $2L$

$\Rightarrow$ $\sqrt{\lambda} L - (-\sqrt{\lambda} L) = 2\sqrt{\lambda} L$ needs to be a multiple of 2π

result: $\sqrt{\lambda} L = n(2\pi) = 2n\pi \quad \Rightarrow \quad \boxed{\sqrt{\lambda} = \frac{2n\pi}{L}}$

Triple Integrals.

we parametrized ball of radius 2 : $x^2+y^2+z^2 \leq 4$
as an elementary region

poss. values for x: $\boxed{-2 \leq x \leq 2}$

given x.. poss. y-values: $\boxed{-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}}$

given x,y: " z-values $\boxed{-\sqrt{4-x^2-y^2} \leq z \leq \sqrt{4-x^2-y^2}}$

calculate volume of ball:

$$vol(B) = \iiint_B 1 \cdot dV$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} 1 \, dz \, dy \, dx = \int\int z \Big|_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dy dx$$